# SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

## 2007

## HIGHER SCHOOL CERTIFICATE ASSESSMENT TASK #3

# Mathematics

**(2 Unit)**

**General Instructions**

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question.
- Hand in your answers in 3 separate bundles. Section A (Question 1 and Question 2), Section B (Question 3 and Question 4) and Section C (Question 5 and Question 6)

**Total Marks – 100**

- Attempt questions 1-6
- All questions are **NOT** of equal value.

Examiner: *E. Choy*

This page is intentionally left blank

## Section A – Start a new booklet.

**Question 1 (23 marks).** **Marks**

**a)** Differentiate with respect to $x$

(i) $\ln 3x$ 2

(ii) $\tan\dfrac{x}{2}$ 2

(iii) $\dfrac{1}{2}\sin 4x$ 2

(iv) $4+5e^{-x}$ 2

**b)** (i) What is the second derivative of $e^{0.2x}$? 2

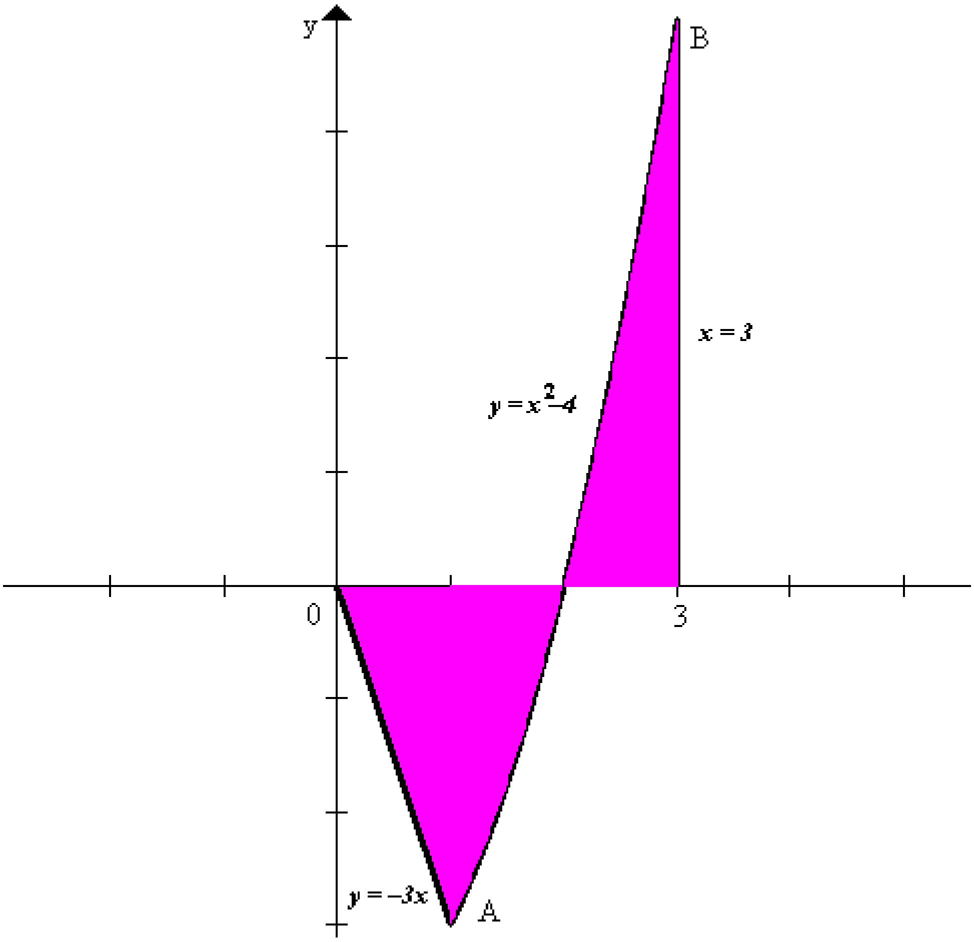
(ii) Find $f''(2)$ if $f(x)=5-2\ln x$ 2

(iii) Find the domain for which $y=x-2x^3$ is concave downwards. 2

(iv) Find the point of inflexion on the curve $y=3x^3-3x^2+3x-1$. 2

**c)** Evaluate $\displaystyle\int_{-1}^{0}\sin\pi x\,dx$ 2

**d)**




The shaded region is bounded by the lines $x = 3$, $y = -3x$, the $x$-axis and the curve $y = x^2 - 4$.

Show that A is the point (1,-3) and find the area of the shaded region of the graph. 3

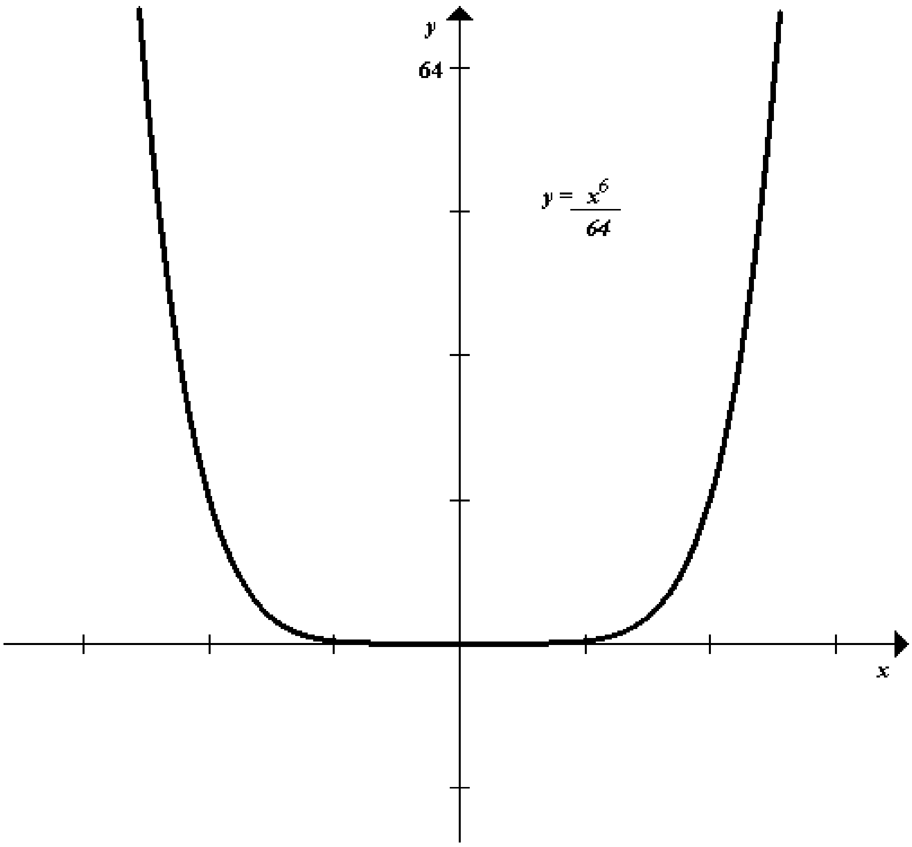
**e)** Sketch the graph of $y = \tan 2x$ for $0 \le x \le 2\pi$ 2

**Question 2 (15 Marks).** **Marks**

**a)** Solve $\log_7 64 = 3\log_7 x$ 1

**b)** Find the exact value of $\int_{\frac{1}{3}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right) dx$ 2

**c)**



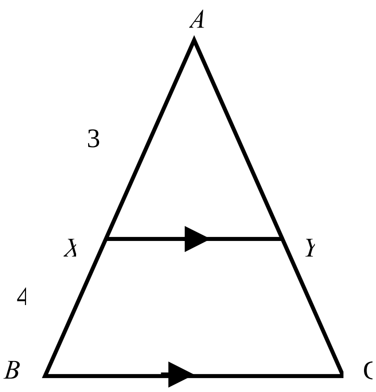

A bowl is formed by rotating part of the curve $y = \frac{x^6}{64}$ between $x = 0$ and $x = 4$ about the $y$-axis.

(i) Show that $x^2 = 4y^{\frac{1}{3}}$ 1

(ii) Find the volume of the bowl. 3

**d)** Convert $240^\circ$ to radians. 1

**e)**




With respect to the diagram above write down the ratios:

(i) $AY:YC$ 1

(ii) $AY:AC$ 1

(iii) $YC:AC$ 1

**f)** $ABC$ is a triangle in which $AB=AC$. If $P,Q$ and $R$ are collinear points on $AC,\ CB$ and $AB$ produced respectively, show that $BR\times PQ=PC\times RQ$ 4

**End of Section A.**

## Section B – Start a new booklet.

**Question 3 (19 marks).** **Marks**

**a)** (i) Copy and complete the following table of values for the curve

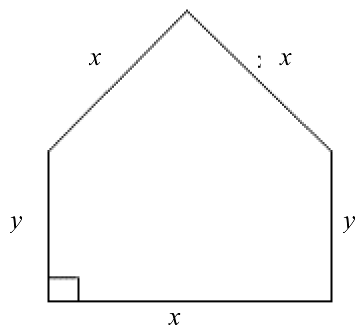
$y = \dfrac{1}{x+1}$ 1

| $x$ | 0 | ½ | 1 | 1½ | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

(ii) Use the trapezoidal rule with 5 function values from part (i), to give an approximation to $\int_0^2 \dfrac{dx}{1+x}$. Give your answer correct to 2 decimal places. 3

**b)** Given that $n$ is a positive number, find the smallest and largest of the following numbers; $e^{\frac{n}{2}}, e^{n}, e^{-n}, e^{-\frac{1}{2}n}$. 1

**c)**

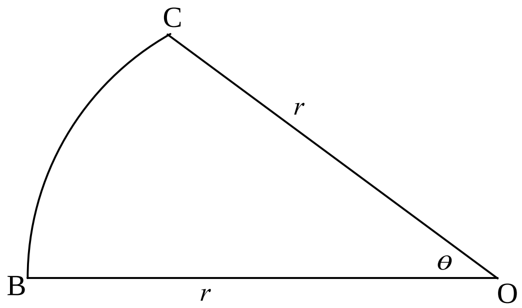



A piece of wire is bent to form a pentagon as shown. The area enclosed by the wire is $33\text{cm}^2$.

(i) Express $y$ in terms of $x$. 2

(ii) Show that the perimeter is a minimum when $x = 2\sqrt{6+\sqrt{3}}$ 4

**d)**




The diagram above shows a sector $OBC$ of a circle with centre $O$ and radius $r$ cm. The arc $BC$ subtends an angle $\theta$ radians at $O$.

(i) Show that the perimeter of the sector is $r\left(2+\theta\right)$. **2**

(ii) Given that the perimeter of the sector is 36cm, show that its area **2**

$$A=\frac{648\theta}{\left(\theta+2\right)^2}$$

(iii) Hence show that the maximum area of the sector is 81cm$^2$. **4**

**Question 4 (14 marks).** **Marks**

**a)** Differentiate $x^2 \ln x$ and hence determine $\int_{\sqrt{e}}^{e} x \ln x \, dx$ 4

**b)** The region $R$ is bounded by the curve $y = \tan x$, the $x$-axis and the vertical line $x = \frac{\pi}{3}$

(i) Sketch $R$ and find its area. 2

(ii) Find the volume generated when $R$ is rotated about the $x$-axis. 2

**c)** Consider the function $y = 1 + e^{2x}$

(i) What is the domain of the function? 1

(ii) Show that $x = \frac{1}{2}\ln(y-1)$ 1

(iii) The volume $V$ formed when the area between $y = 1 + e^{2x}$, the $y$-axis and the lines $y = 2$ and $y = 6$, is rotated about the $y$-axis is given by $V = \frac{\pi}{4}\int_{2}^{6} \left[\ln(y-1)\right]^2 dy$.

Use Simpson's rule with 5 function values to estimate this volume.
Leave your answer rounded to 3 significant figures. 4

**End of Section B.**

## Section C – Start a new booklet.

**Question 5 (15 marks).** **Marks**

**a)** Richard won the NSW Lotto on the 1st January 2000. The prize was one million dollars. He decided to deposit the entire amount into an account which paid interest at a rate of 8% per annum. The interest was calculated quarterly and compounded quarterly. Richard then made an annual withdrawal $50000, starting on the 1st January 2001.

(i) Write down an expression for the amount in Richard's account immediately after his first withdrawal. 2

(ii) Show that the amount in Richard's account immediately after his third withdrawal is given by the expression: 3

$$10^6 \times 1.02^{12} - 50000\left(1 + 1.02^4 + 1.08^8\right)$$

(iii) How much, to the nearest dollar, was left in Richard's account after his twentieth withdrawal. 1

**b)** Given that $f(x) = x\left(x-2\right)^2$

(i) Show that $f'(x) = 3x^2 - 8x + 4$ 1

(ii) Find 2 values of x for which $f'(x) = 0$ and give corresponding values of $f(x)$. 2

(iii) Determine the nature of the turning points of the curve $y = f(x)$. 2

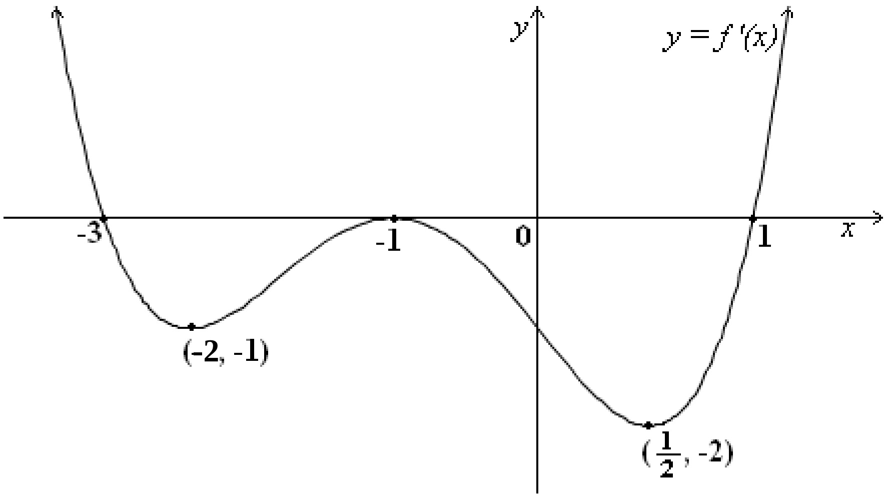
(iv) Find where the curve $y = f(x)$ crosses the $x$-axis. 1

(v) Sketch the curve $y = f(x)$. 2

(vi) Use your sketch to solve the inequation $x\left(x-2\right)^2 \geq 0$ 1

**Question 6 (14 marks).** **Marks**

**a)**




Consider the graph of $y = f'(x)$ shown above. Find the x co-ordinates of the minimum turning point and the maximum turning point of the graph $y = f(x)$. 3

**b)** Consider the two functions $f(x)$ and $g(x)$ where:

$$f(x) = \frac{e^x - e^{-x}}{2} \qquad g(x) = \frac{e^x + e^{-x}}{2}$$

(i) Show that $f'(x) = g(x)$ and $g'(x) = f(x)$. 2

(ii) Show that the graph $y = f(x)$ is increasing for all values of $x$ and that there is a point of inflexion at $x = 0$. 2

(iii) Show that the graph $y = g(x)$ has a minimum at $x = 0$. 2

(iv) Sketch the graph of $y = g(x)$. 2

(v) Let $y = f(x)$. Show that this equation can be written in the form:

$$e^{2x} - 2ye^x - 1 = 0$$ 3

Hence deduce that $x = \ln\left(y + \sqrt{y^2 + 1}\right)$.

**End of Section C.**

**End of Examination.**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

1) a) i) $y = \ln 3x$

$y' = \frac{3}{3x}$

$y' = \frac{1}{x}$

ii) $y = \tan \frac{x}{2}$

$y' = \frac{1}{2} \sec^2 \frac{x}{2}$

iii) $y = \frac{1}{2} \sin 4x$

$y' = 2 \cos 4x$

iv) $y = 4 + 5e^{-x}$

$y' = -5e^{-x}$

b) i) $y = e^{0.2x}$

$y' = 0.2 e^{0.2x}$

$y'' = 0.04 e^{0.2x}$

ii) $f(x) = 5 - 2 \ln x$

$f'(x) = -\frac{2}{x}$

$= -2x^{-1}$

$f''(x) = 2x^{-2}$

$= \frac{2}{x^2}$

$f''(2) = \frac{2}{(2)^2}$

$= \frac{1}{2}$

iii) $y = x - 2x^3$

$y' = 1 - 6x^2$

$y'' = -12x$

$y'' < 0$

$-12x < 0$

$x > 0$

iv) $y = 3x^3 - 3x^2 + 3x - 1$

$y' = 9x^2 - 6x + 3$

$y'' = 18x - 6$

let $y'' = 0$

$18x - 6 = 0$

$18x = 6$

$x = \frac{1}{3}$

when $x = \frac{1}{3}$

$y = 3(\frac{1}{3})^3 - 3(\frac{1}{3})^2 + 3(\frac{1}{3}) - 1$

$y = -\frac{2}{9}$

The point of inflexion is at $(\frac{1}{3}, -\frac{2}{9})$

c) $\int_{-1}^{0} \sin \pi x \, dx$

$= \left[ -\frac{1}{\pi} \cos \pi x \right]_{-1}^{0}$

$= -\frac{1}{\pi} \cos(0) - \left( -\frac{1}{\pi} \cos(-\pi) \right)$

$= -\frac{1}{\pi} - \frac{1}{\pi}$

$= -\frac{2}{\pi}$

d) $y = x^2 - 4$ ——— 1

$y = -3x$ ——— 2

sub 1 into 2

$x^2 - 4 = -3x$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4, \quad x = 1$

clearly $x = 1$

when $x = 1$

$y = -3(1)$

$y = -3$

$\therefore A$ is the point $(1, -3)$

$$\text{Area} = -\int_0^1 -3x\,dx - \int_1^2 (x^2-4)\,dx + \int_2^3 (x^2-4)\,dx$$

$$= \int_0^1 3x\,dx + \int_1^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx$$

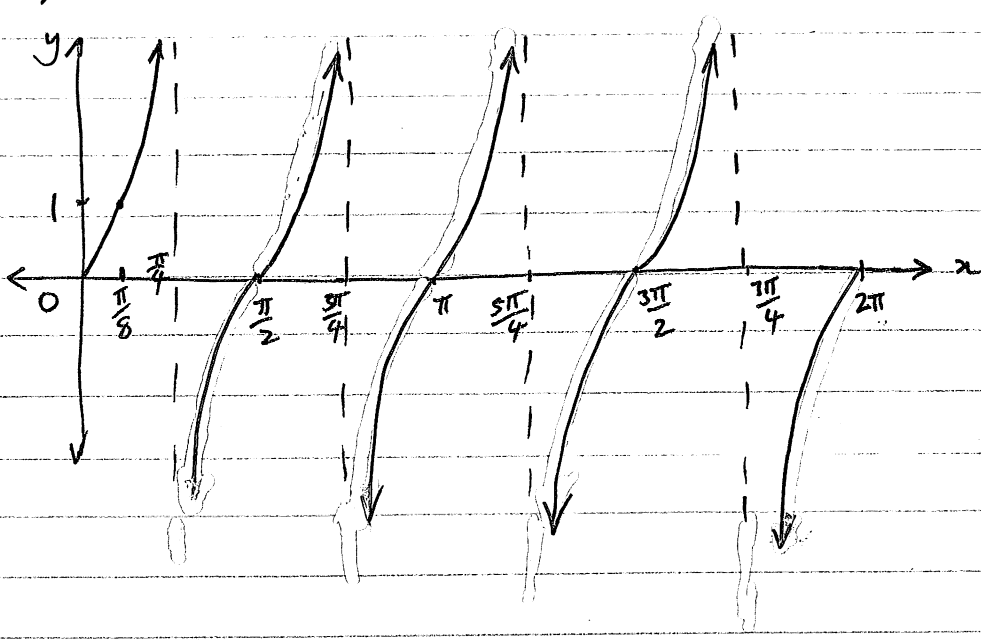
$$= \left[\frac{3x^2}{2}\right]_0^1 + \left[4x - \frac{x^3}{3}\right]_1^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= \left[\frac{3(1)^2}{2} - 0\right] + \left[4(2) - \frac{(2)^3}{3} - \left(4(1) - \frac{(1)^3}{3}\right)\right] + \left[\frac{(3)^3}{3} - 4(3) - \left(\frac{(2)^3}{3} - 4(2)\right)\right]$$

$$= \frac{11}{2} \text{ units}^2$$

e) $y = \tan 2x$

period $= \frac{\pi}{2}$



2) a) $\log_7 64 = 3\log_7 x$

$\log_7 64 = \log_7 x^3$

$\therefore x^3 = 64$

$x = 4$

b) $\int_{\frac{1}{3}}^{\frac{1}{2}} \sec^2\left(\frac{\pi x}{2}\right) dx$

$= \left[\frac{1}{\left(\frac{\pi}{2}\right)} \tan\left(\frac{\pi x}{2}\right)\right]_{\frac{1}{3}}^{\frac{1}{2}}$

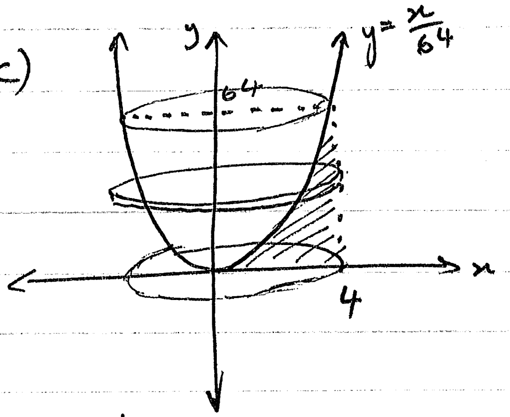
$= \left[\frac{2}{\pi} \tan\left(\frac{\pi x}{2}\right)\right]_{\frac{1}{3}}^{\frac{1}{2}}$

$= \frac{2}{\pi}\tan\left(\frac{\pi}{4}\right) - \frac{2}{\pi}\tan\left(\frac{\pi}{6}\right)$

$= \frac{2}{\pi}(1) - \frac{2}{\pi}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{2}{\pi}\left(1 - \frac{1}{\sqrt{3}}\right)$

c)




when $x = 4$

$y = \frac{(4)^6}{64}$

$y = 64$

i) $y = \frac{x^6}{64}$

$x^6 = 64y$

$x^2 = \sqrt[3]{64y}$

$x^2 = 4y^{\frac{1}{3}}$

ii) $V = \pi \int_a^b x^2 \, dy$

$V = \pi \int_0^{64} (4)^2 dy - \pi \int_0^{64} 4y^{\frac{1}{3}} dy$

$= 16\pi \int_0^{64} dy - 4\pi \int_0^{64} y^{\frac{1}{3}} dy$

$= 16\pi \left[y\right]_0^{64} - 4\pi \left[\frac{3}{4} y^{\frac{4}{3}}\right]_0^{64}$

$= 16\pi(64 - 0) - 3\pi\left((64)^{\frac{4}{3}} - 0\right)$

$= 256\pi$ units$^3$

d) $240^\circ \times \frac{\pi}{180} = \frac{4\pi}{3}$

e) i) $AY : YC = AX : XB$

$= 3 : 4$

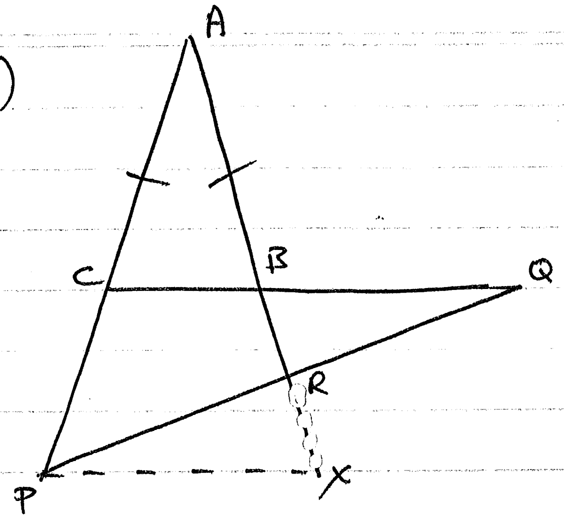
ii) $AY : AC = AX : AB$

$= 3 : 7$

iii) $YC : AC = XB : AB$

$= 4 : 7$

f)




X lies on AR produced such that $PX // CB$.

In $\triangle$'s BRQ & PRX

$\angle BRQ = \angle PRX$ (vert. opp. $\angle$'s)

$\angle BQR = \angle RPX$ (alt $\angle$'s $BQ // PX$)

$\therefore$ BRQ $|||\triangle$ PRX (equiangular)

$\therefore \dfrac{BR}{RQ} = \dfrac{RX}{PR}$ (corr. sides in same ratio)

$BR.PR = RX.RQ$

$AB = AC$ (given)

clearly $AX = AP$

$\therefore PC = BX$

$= BR + RX$

$\therefore RX = PC - BR$

$BR.PR = (PC - BR)RQ$

$BR.PR = PC.RQ - BR.RQ$

$BR.PR + BR.RQ = PC.RQ$

$BR(PR + RQ) = PC.RQ$

$BR.PQ = PC.RQ.$

Q3 (a) (i)

| $x$ | 0 | $1/2$ | 1 | $1\frac{1}{2}$ | 2 |
|---|---|---|---|---|---|
| $y$ | 1 | $2/3$ | $1/2$ | $2/5$ | $1/3$ |

(1)

(ii) $\frac{\frac{1}{2}}{2}\left[1+\frac{1}{3}+2\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right)\right]$

$= \frac{1}{4}\left[\frac{4}{3}+2\left(\frac{20+15+12}{30}\right)\right]$

$= \frac{1}{4}\left[\frac{4}{3}+\frac{47}{15}\right]$

$= \frac{1}{4}\,\frac{20+47}{15}$

(3)

$= \frac{67}{60}$

$= \underline{1.12}$

(b) Smallest $e^{-n}$

Largest $e^{n}$.

(1)

(c) (i) $A = xy + \frac{1}{2}x^2\sin 60°$

$= xy + \frac{1}{2}x^2 \cdot \frac{\sqrt{3}}{2}$

(2)

$= xy + \frac{\sqrt{3}x^2}{4} = 33$

$\therefore xy = 33 - \frac{\sqrt{3}x^2}{4}$

$y = 33x^{-1} - \frac{\sqrt{3}}{4}x.$

(ii) $P = 3x + 2y$

$= 3x + 66x^{-1} - \frac{\sqrt{3}}{2}x.$

$P' = 3 - \frac{\sqrt{3}}{2} - 66x^{-2}$

$P'' = 132x^{-3}$

Let $P' = 0$ for max/min.

$3 - \frac{\sqrt{3}}{2} = \frac{66}{x^2}$.

$6 - \sqrt{3} = \frac{132}{x^2}$

$\frac{x^2}{132} = \frac{1}{6-\sqrt{3}}$

$= \frac{6+\sqrt{3}}{33}$

$\therefore x^2 = \frac{132(6+\sqrt{3})}{33}$

$= 4(6+\sqrt{3})$

$x = \pm 2\sqrt{6+\sqrt{3}}$ ✓✓

Clearly $x > 0$ & a min occurs as $P'' > 0$ ✓

at $x = 2\sqrt{6+\sqrt{3}}$

(d). (i) $P = r + r + r\theta$

$= 2r + r\theta$

$= r(2+\theta)$. ✓✓

(ii) If $P = 36$.

$r(2+\theta) = 36$.

$r = \frac{36}{2+\theta}$

now $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \cdot \left(\frac{36}{2+\theta}\right)^2 \cdot \theta$.

$\boxed{A = \frac{648\theta}{(2+\theta)^2}}$ ✓✓

(iii) $A = \dfrac{648\theta}{(2+\theta)^2}$

$$A' = \frac{(2+\theta)^2 . 648 - 648\theta . 2(2+\theta)}{(2+\theta)^4}$$

$$= \frac{648(2+\theta)\left[2+\theta-2\theta\right]}{(2+\theta)^4}$$

$$= \frac{648(2-\theta)}{(2+\theta)^3}.$$

✓✓

Let $A' = 0$

$$\frac{648(2-\theta)}{(2+\theta)^3} = 0$$

$$\theta = 2.$$

$$\therefore A = \frac{648 \times 2}{4^2} = \frac{648}{8} = 81.$$ ✓

| $\theta$ | 1 | 2 | 3 |
|---|---|---|---|
| $A'$ | $\frac{648}{3^3}$ $>0$ | 0 | $-\frac{648}{5^3}$ $<0$ |
| | / | — | \ |

$\therefore$ MAX.

✓

NB The "2nd derivative test" is probably not the best option.

also. / — \ is
+ 0 −

not sufficient (we were told it is a max) & you need to show it!!

Q4. a

$$\frac{d}{dx}(x^2 \ln x) = 2x \ln x + x^2 \times \frac{1}{x}$$
$$= 2x \ln x + x.$$

$$\therefore \int_{\sqrt{e}}^{e} (2x \ln x + x)\,dx = \left[x^2 \ln x\right]_{\sqrt{e}}^{e}$$
$$= e^2 \ln e - (\sqrt{e})^2 \ln\sqrt{e}$$
$$= e^2 - e \cdot \times \frac{1}{2}$$
$$= e^2 - \frac{e}{2}.$$

$$\therefore \int_{\sqrt{e}}^{e} 2x \ln x\,dx + \int_{\sqrt{e}}^{e} x\,dx = e^2 - \frac{e}{2}$$

$$2\int_{\sqrt{e}}^{e} x \ln x\,dx + \left[\frac{x^2}{2}\right]_{\sqrt{e}}^{e} = e^2 - \frac{e}{2}$$

$$2\int_{\sqrt{e}}^{e} x \ln x\,dx + \frac{e^2}{2} - \frac{(\sqrt{e})^2}{2} = e^2 - \frac{e}{2}$$

$$2\int_{\sqrt{e}}^{e} x \ln x\,dx + \frac{e^2}{2} - \frac{e}{2} = e^2 - \frac{e}{2}$$

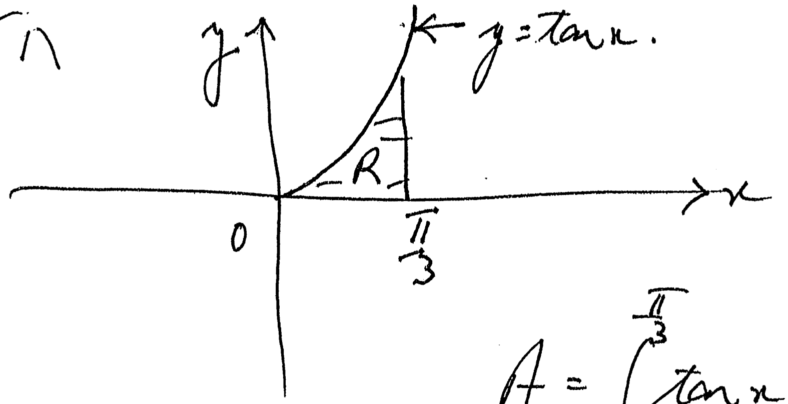
$$2\int_{\sqrt{e}}^{e} x \ln x\,dx = \frac{e^2}{2}$$

$$\therefore \int_{\sqrt{e}}^{e} x \ln x\,dx = \frac{e^2}{4}$$

Q4 b

(CONT)

(i)




$$A = \int_0^{\frac{\pi}{3}} \tan x \, dx.$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx.$$

$$= -\Big[\ln(\cos x)\Big]_0^{\frac{\pi}{3}}$$

$$= -\Big[\ln \tfrac{1}{2} - \ln 1\Big]$$

$$= \underline{\ln 2} \quad u^2$$

(ii) $$V = \pi \int_0^{\frac{\pi}{3}} \tan^2 x \, dx.$$

$$= \pi \int_0^{\frac{\pi}{3}} (\sec^2 x - 1) \, dx$$

$$= \pi \Big[ \tan x - x \Big]_0^{\frac{\pi}{3}}$$

$$= \pi \Big[ \tan \tfrac{\pi}{3} - \tfrac{\pi}{3} - (0 - 0) \Big]$$

$$= \underline{\pi \cdot \left( \sqrt{3} - \tfrac{\pi}{3} \right) u^3.}$$

Q4 (CONTD)

c (i) $x \in \mathbb{R}$. (all Reals etc)

(ii) $y = 1 + e^{2x}$

$y - 1 = e^{2x}$.

$\therefore 2x = \ln(y-1)$

$\boxed{x = \frac{1}{2}\ln(y-1)}$

(iii) Using Simpson's Rule to evaluate the integral

$$\frac{\pi}{4}\int_2^6 [\ln(y-1)]^2 \, dy. \quad \left(\text{let } f(y) = [\ln(y-1)]^2\right)$$

| $y$ | 2 | 3 | 4 | 5 | 6. |
|---|---|---|---|---|---|
| $f(y)$ | $(\ln 1)^2 = 0$ | $(\ln 2)^2 = 0.4804$ | $(\ln 3)^2 = 1.2069$ | $(\ln 4)^2 = 1.9218$ | $(\ln 5)^2 = 2.5902$ |

$$\therefore V = \frac{\pi}{4} \times \frac{1}{3}\Big[0 + 2.5902 + 4(0.4804 + 1.9218) + 2(1.2069)\Big]$$

$\doteqdot 3.83 \; u^3$ (to 3. SIG FIGS)

Task 3 2007 2 unit Solutions.

5 (a) January $1^{st}$ 2000 \$1,000,000

$r = 8\%$ p.a. $\Rightarrow \frac{8}{4} = 2\%$ per quarter

interest calculated quarterly, paid quarterly.

Annual withdrawal: \$50,000 starting Jan 1 2001.

(i) Jan 1 2001 $A_1 = 1{,}000{,}000\left(1+\frac{2}{100}\right)^4 - 50\,000$

$= 10^6 \times 1.02^4 - 50{,}000$ 2

(ii) Jan 1 2002 $A_2 = (10^6 \times 1.02^4 - 50{,}000)\left(1+\frac{2}{100}\right)^4 - 50000$

$= (10^6 \times 1.02^4 - 50000)(1.02)^4 - 50000$

$= 10^6 \times 1.02^8 - 50000 - 50000 \times 1.02^4$

$= 10^6 \times 1.02^8 - 50000(1 + 1.02^4)$.

$\therefore$ Jan 1 2003 $A_3 = 10^6 \times 1.02^{12} - 50000(1 + 1.02^4 + 1.02^8)$.

3

(iii) $20^{th}$ withdrawal

$A_{20} = 10^6 \times 1.02^{80} - 50000(1 + 1.02^4 + 1.02^8 + \cdots + 1.02^{76})$

using $\frac{a - rl}{1-r}$

$= 10^6 \times 1.02^{80} - 50000\left(\frac{1 - 1.02^4 \times 1.02^{76}}{1 - 1.02^4}\right)$

$= 1{,}000{,}000 \times 1.02^{80} - 50{,}000\left(\frac{1 - 1.02^{80}}{1 - 1.02^4}\right)$

$= 4875439.156 - 2350683.978$

$= \$2524755.18$

$\doteqdot \$2524755$ was left. 1

# Sydney Boys' High School

Student No.: ______________________

______________________

Paper: ______________________

______________________

Section: __________

Sheet No.: __________ of __________ for this Section.

| Q.No | Tick | Mark |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

5 (b) $f(x) = x(x-2)^2$

(i) $f(x) = x(x-2)(x-2)$

$= x(x^2 - 4x + 4)$

$= x^3 - 4x^2 + 4x$

$f'(x) = 3x^2 - 8x + 4$ 1

(ii) When $3x^2 - 8x + 4 = 0$

$\times\ 12$
$+\ -8$

$\dfrac{(3x-6)(3x-2)}{3} = 0$

$\dfrac{\cancel{3}(x-2)(3x-2)}{\cancel{3}} = 0$

$x = 2 \qquad x = 2/3$

$f(x) = 2(2-2)^2 = 0 \qquad f(x) = \frac{2}{3}\left(\frac{2}{3} - 2\right)^2$

$= \frac{2}{3} \times -\frac{4}{3} \times -\frac{4}{3} = 1\frac{5}{27}$

$(2, 0)$ 1 $\qquad \left(\frac{2}{3}, 1\frac{5}{27}\right)$ 1

(iii) now $f'(x) = 3x^2 - 8x + 4$

$f''(x) = 6x - 8$

at $(2, 0)$ $f''(x) = 12 - 8 = 4 > 0$ min t. pt 1

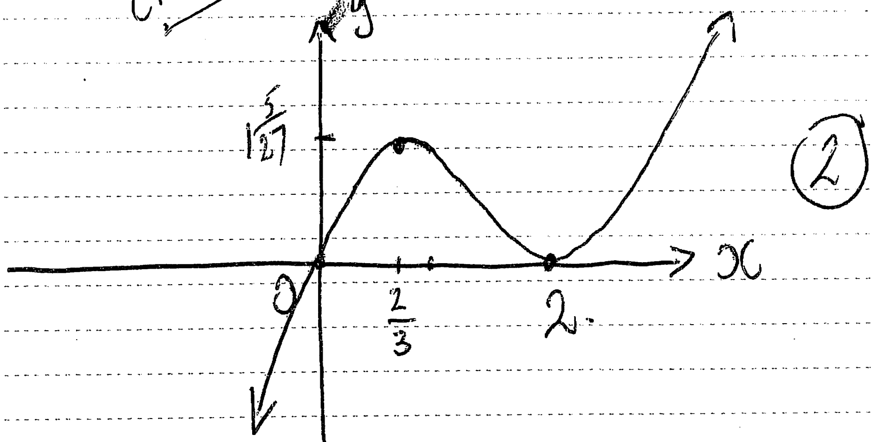
at $(\frac{2}{3}, 1\frac{5}{27})$ $f''(x) = 6 \times \frac{2}{3} - 8 = 4 - 8 = -4 < 0$

MAX t. pt 1

(iv) $f(x) = x(x-2)^2$

when $f(x) = 0$ crosses $x = 0$, ~~$x = 2$ (double root)~~ touch 1

(v)




check, $x = 3$ $f(x) = 3(3-2)^2 = 3$

$x = -1$, $f(x) = -1(-1-2)^2 = -1 \times 9 = -9$

(vi) $x(x-2)^2 \geq 0$ $\Rightarrow$ $x \geq 0$ from sketch

1

Sydney Boys' High School

Student No.: ______________________

Paper: ______________________

Section: ________

Sheet No.: ________ of ________ for this Section.

| Q.No | Tick | Mark |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

Q6 (a) $f'(x)$ gives gradient of tangent lines at $x$ values eg $(-3,-1)$ means at $x=-3$ on the original curve, the slope of the tangent line at $x=-3$ is $m=-1$.

minimum turning pt $x-\varepsilon$ \ negative slope

$x$ 0

$x+\varepsilon$ / positive slop

$x=1-\varepsilon$ slope negative

$x=1$ slope is zero $1\frac{1}{2}$

$x=1+\varepsilon$ slope positive

MAX turning point $x-\varepsilon$ / positive slope

$x$ 0

$x+\varepsilon$ \ negative slope

$1\frac{1}{2}$

$x=-3-\varepsilon$ slope +ve

$x=-3$ slope 0

$x=-3+\varepsilon$ slope −ve

6 (b) $f(x) = \frac{e^{x} - e^{-x}}{2}$ $g(x) = \frac{e^{x} + e^{-x}}{2}$

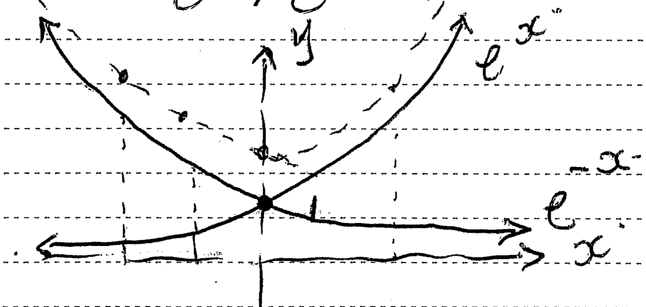
(i) $f'(x) = \frac{e^{x} - e^{-x} \times -1}{2} = \frac{e^{x} + e^{-x}}{2} = g(x)$

and. $g'(x) = \frac{e^{x} + e^{-x} \times -1}{2} = \frac{e^{x} - e^{-x}}{2} = f(x)$. 2

(ii) If $y = f(x)$ is an increasing graph then $f'(x) > 0 \ \forall x$

show $\frac{e^{x} + e^{-x}}{2} > 0 \ \forall x$

show $e^{x} + e^{-x} > 0$




The addition of ordinates of $e^{x}, e^{-x} \ \forall x$ will always be $> 0$. 1

$f'(x) = \frac{e^{x} + e^{-x}}{2}$

$f''(x) = \frac{e^{x} + e^{-x} \times -1}{2}$

$= \frac{e^{x} - e^{-x}}{2}$

So at $x = 0 - \varepsilon$ (eg $-\cdot 1$) $f''(x) = \frac{e^{-\cdot 1} - e^{\cdot 1}}{2} < 0$

$x = 0$ $f''(x) = \frac{1-1}{2} = 0$

$x = 0 + \varepsilon$ (eg $\cdot 1$) $f''(x) = \frac{e^{\cdot 1} - e^{-\cdot 1}}{2} > 0$

sign change

Inflexion at $x = 0$. 1

(iii) $g(x) = \dfrac{e^{x} + e^{-x}}{2}$

$g'(x) = \dfrac{e^{x} - e^{-x}}{2}$

When $\dfrac{e^{x} - e^{-x}}{2} = 0$

$\dfrac{e^{x}}{1} - \dfrac{1}{e^{x}} = 0$

$\dfrac{e^{x}}{1} = \dfrac{1}{e^{x}}$

$(e^{x})^{2} = 1$

$e^{2x} = e^{0}$

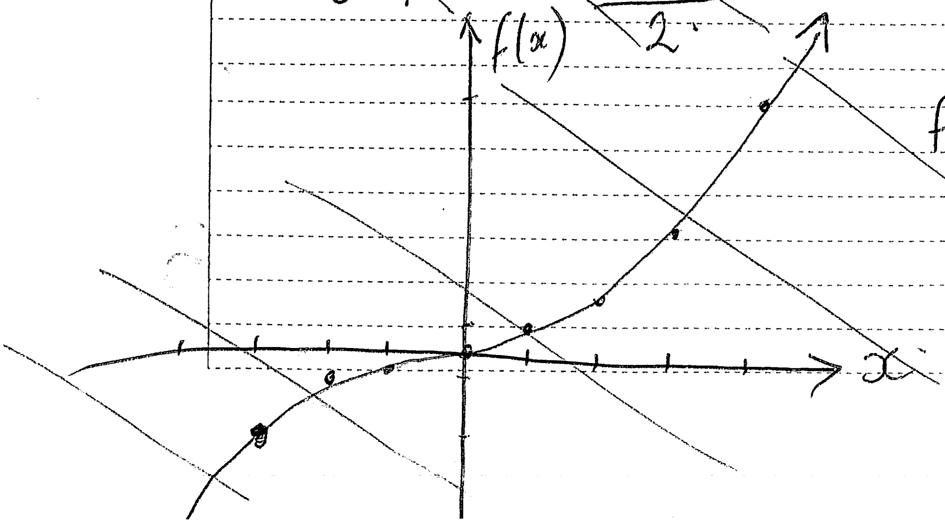
$2x = 0 \Rightarrow x = 0$

$g''(x) = \dfrac{e^{x} + e^{-x}}{2}$

2

at $x=0$, $g''(x) = \dfrac{e^{0} + e^{-0}}{2} = \dfrac{1+1}{2} > 0$ min

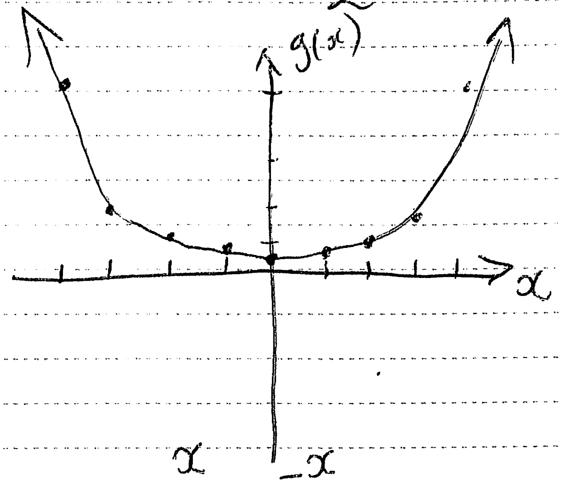
(iv) $y = f(x) = \dfrac{e^{x} - e^{-x}}{2}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$: | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| $f(x)$: | -27.3 | -10 | -3.6 | -1.2 | 0 | 1.2 | 3.6 | 10 | 27.3 |

1

$y = g(x) = \dfrac{e^{x} + e^{-x}}{2}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$: | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| $g(x)$: | 27.3 | 10 | 3.8 | 1.5 | 1 | 1.5 | 3.8 | 10 | 27.3 |

2

(iv) $y = \dfrac{e^{x} - e^{-x}}{2}$

$2y = e^{x} - \dfrac{1}{e^{x}}$

$2ye^{x} = e^{2x} - 1$

$e^{2x} - 2ye^{x} - 1 = 0$

now for $e^{2x} - 2ye^{x} - 1 = 0$

let $w = e^{x}$

so $w^2 - 2wy - 1 = 0$

$a = 1,\ b = -2y,\ c = -1$

$w = \dfrac{2y \pm \sqrt{4y^2 - 4 \times 1 \times -1}}{2}$

$= \dfrac{2y \pm \sqrt{4y^2 + 4}}{2}$

$= \dfrac{2y \pm 2\sqrt{y^2+1}}{2}$

$w = y \pm \sqrt{y^2+1}$

now $e^{x} = y \pm \sqrt{y^2+1}$

$\ln e^{x} = \ln\left(y \pm \sqrt{y^2+1}\right)$

$x = \ln\left(y \pm \sqrt{y^2+1}\right)$

y can be any value so for ln to exist $y + \sqrt{y^2+1} \geq 0$ is chosen.

So $x = \ln\left(y + \sqrt{y^2+1}\right)$

3